411 Individual HW3

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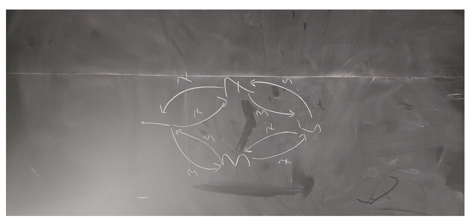
October 2023

Problem 1

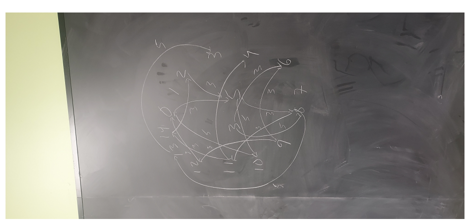
1. $\mathbb{Z}_6$ has 1 and 5 as generators.
2. $\mathbb{Z}_1 1^*$ has generators 2, 6, 7, 8.
3. $\mathbb{Z}_1 4^*$ has generators 3 and 5.

Problem 2

1. $\mathbb{Z}_8{}^*$ is generated by $\{3, 7\}$, $3 \cdot 7 = 5$, $7 \cdot 7 = 1$.



2. $\mathbb{Z}_{15}{}^*$ is generated by $\{3, 5\}$.



Problem 4

a) Assume that we have A, B in $2^{\{1,2,3\}}$ such that A, B generate the group. Call the operation $(A \cup B)/(A \cap B)$, $\cdot$. We observe that since intersection and union are commmutative that this operation is also commutative. From the last homework we know that for any set C, we have that $C \cdot C$ is $\emptyset$. So A, B can then generate the null set. For $A \cdot B$ we have $(A \cap B)/(A \cup B)$. We observe that we have now generated 4 elements $S = \{\emptyset, A, B, (A \cap B)/(A \cup B)\}$, and will show that we cannot generate any more. We can check all pairs of elements in the set and can do this in 10 tries due to commutativity. $\emptyset \cdot \emptyset = \emptyset \in S, \emptyset \cdot A = A \in S, \emptyset \cdot B = B \in S, \emptyset \cdot (A \cup B)/(A \cap B) = (A \cup B)/(A \cap B) \in S, A \cdot A = \emptyset \in S, A \cdot B = (A \cup B)/(A \cap B) \in S, B \cdot B = \emptyset \in S$. The final two calculations, $A \cdot (A \cup B)/(A \cap B)$, and $B \cdot (A \cup B)/(A \cap B)$ will be handled with more care.

$$A \cdot (A \cup B)/(A \cap B)$$

$$(A \cup ((A \cup B)/(A \cap B)))/(A \cap ((A \cup B)/(A \cap B)))$$

We observe that $(A \cup ((A \cup B)/(A \cap B)))$ is equal to $A \cup B$, because $(A \cap B) \subset A$. We also see that the second half is the intersection of those elements which are exclusively in A or B with A. We see that all elements which are exclusively in A are in this intersection, and none of the elements which are exclusively in B are in it. So this whole expression simplifies to

$$(A \cup B)/(A/B)$$

. We will show that this is equal to B by showing that both sets are contained in eachother. We rewrite $(A \cup B)/(A/B)$ as $(B/A) \cup (B \cap A) \cup (A/B)$. Then $(A \cup B)/(A/B)$ is $(B/A) \cup (B \cap A)$, or those elements which are exclusively in B and the elements which are in B and A. This clearly simplifies to B. A similar calculation for B follow which shows that $B \cdot (A \cup B)/(A \cap B) = A$. So we can only generate 4 elements with 2 subsets. Since 4 is less than the order of the group which is 8, it cannot be generated.

b) We see that the group is generated by $\{\{1\}, \{2\}, \{3\}\}$. $\{1\} \cdot \{1\} = \emptyset$, $\{1\} \cdot \{2\} = \{1, 2\}$, $\{1\} \cdot \{3\} = \{1, 3\}$, $\{2\} \cdot \{3\} = \{2, 3\}$, $\{1\} \cdot \{2, 3\} = \{1, 2, 3\}$.

Problem 5

https://www.wolframcloud.com/obj/401dfafe-3b70-4303-bb4b-6676c4e18e5c

Problem 8

So we can map the elements of S_5 to some categories, namely the partitions of the number 5:

$$1 + 1 + 1 + 1 + 1$$

$$2 + 1 + 1 + 1$$

$$2 + 2 + 1$$

$$3 + 1 + 1$$
$$3 + 2$$
$$4 + 1$$
$$5 + 0$$

where each number in each partition represents a cycle of that length. 1's simply represent "1-cycles" or elements which sit still. We can see that the two of these types of partitions which do not represent permutations that are cycles are:

$$3 + 2 = (abc)(de)$$
$$2 + 2 + 1 = (ab)(cd)$$

so let's count how many of these exist. In the case of $(abc)(de)$ there are 5 ways to choose a, 4 ways to choose b and so on. We also notice that for the first permutation $(abc) = (bca) = (cab)$ and for the second permutation $(de) = (ed)$. So, incorporating in this finding to avoid counting same permutations multiple times there are

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2} = 20$$

permutations of this type. In the case of $(ab)(cd)$ there are again 5 ways to choose a, 4 for b, and so on but to the computation to avoid double counting is a bit different. We see that each permutation can be anchored at 2 places, and also that $(ab)(cd) = (cd)(ab)$. So there are:

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2} = 15$$

permutations of this type. We know that there are 120 permutations in total, and 35 permutations that are not cycles. So there are 85 permutations which are cycles.

Problem 9

Assume we have a finite set $S = \{q_1, q_2, q_3, \cdots q_n\}$ which generates $\mathbb{Q}$. We can rewrite this set as $\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_n}{b_n}\}$ where $a_i, b_i \in \mathbb{Z}$ and a_i, b_i are coprime. We see that any element which is generated by S can be written as

$$\sum_{i=1}^{k} \frac{c_i a_i}{b_i}$$

where $c_i \in \mathbb{Z}$ and $c_i \neq 0$, and $\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots, \frac{a_k}{b_k}\} \subset S$. Let's explicitly write this sum out as:

$$\frac{c_1 a_1}{b_1} + \frac{c_2 a_2}{b_2} + \frac{c_3 a_3}{b_3} + \cdots + \frac{c_k a_k}{b_k}$$

. We multiply each term m by $\frac{\prod_{i=1,i\neq m}^{k} b_i}{\prod_{i=1,i\neq m}^{k} b_i}$. This yields:

$$\frac{(b_2b_3b_4\cdots b_k)c_1a_1}{b_1b_2b_3\cdots b_k} + \frac{(b_1b_3b_4\cdots b_k)c_2a_2}{b_1b_2b_3\cdots b_k} + \cdots + \frac{(b_1b_2b_3\cdots b_{k-1})c_ka_k}{b_1b_2b_3\cdots b_k}$$

or just

$$\frac{(b_2b_3b_4\cdots b_k)c_1a_1 + (b_1b_3b_4\cdots b_k)c_2a_2 + \cdots (b_1b_2b_3\cdots b_{k-1})c_ka_k}{b_1b_2b_3\cdots b_k}$$

. We observe the fact that there exists one or more subset of S such that the product $P = \prod_{i=1}^{k} b_i$ is maximal. It is clear that since S is finite that the maximal product P is also finite. But then we cannot generate elements like $\frac{1}{P+1}$ which is clearly in $\mathbb{Q}$. So S cannot be finite.