# 411 Individual HW5

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## Problem 1

a) We observe that the permutation $(x\ y)(z\ t)$ can be written as $\sigma = (x\ y\ z)(y\ z\ t)$. This is because we can see that $\sigma(x) = y, \sigma(y) = z, \sigma(z) = t, \sigma(t) = z$.

b) Since any element in $A_n$ is a product of pairs of transpositions, and each pair of transpositions can be written as a product of 2 3-cycles, all elements in $A_n$ can be generated by some set of 3-cycles.

## Problem 2

a) The permutation with the least disorders is the identity permutation in $S_n$. This is because if we pick any $i$, no pair $(i, j), i < j$ will be a disorder as $\sigma_j = j > i = \sigma_i$ for $\sigma = Id$. Thus the identity permutation has 0 disorders. This must be the smallest number of disorders as you cannot have a "negative" number of disorders.

b) The permutation with the most disorders is $\sigma(k) = n-k+1$, i.e. the "reverse" of the identity map. We will show that $\sigma$ makes every pair $(i, j), i < j$ a disorder. Choose $i$ Then $\sigma_i = n - i + 1$. Choose arbitrary $j > i$. Then:

$$j > i$$

$$-j < -i$$

$$\sigma_j = n + 1 - j < n + 1 - i = \sigma_i$$

Thus, for every pair $(i, j)$ if $\sigma_i > \sigma_j$ and thus every pair is a disorder. Clearly we cannot do better than every pair being a disorder and thus this permutation has the maximum number of disorders.

## Problem 3

In the last homework we showed that $\{(1\ 2), (2\ 3), (3\ 4), \cdots, (n\ n{-}1)\}$ generates $S_n$, so it is sufficient to show that we can create any adjacent transposition from (1 2) and $(1\ 2\ 3\ 4\ \cdots\ n)$. Let us say we want to create the transposition $(k\ k+1) \in S_n$: We contend that this can be written as:

$$(1\ 2\ 3\ 4\ \cdots\ n)^{k-1}(1\ 2)(1\ 2\ 3\ 4\ \cdots\ n)^{n-k+1}$$

Call $(1\ 2\ 3\ 4\ \cdots\ n)$, $\sigma$. We then observe that: $\sigma^{k-1}\sigma^{n-k+1} = \sigma^{k-1+n-k+1} = \sigma^n = Id$. So we can rewrite this as:

$$(1\ 2\ 3\ 4\ \cdots\ n)^{k-1}(1\ 2)(1\ 2\ 3\ 4\ \cdots\ n)^{k-1^{-1}}$$

From the last homework we know that this simply becomes:

$$(\sigma(1), \sigma(2))$$

which is just

$$(k, k+1)$$

Since any permutation can be decomposed into a product of transpositions, which can itself be decomposed into a product of adjacent transpositions, which can then be decomposed into (1 2) and $(1\ 2\ 3\ 4\ \cdots\ n)$, $S_n$ is generated by this set.

# Problem 4

We can break this case down into 4 subcases:

In the first case, we let $\rho_k < \rho_i$ and $\rho_k < \rho_j$: In this case, in $\rho$, $(i, k)$ is a disorder and $(k, j)$ is an order. In $\sigma = \tau\rho$, $(i, k)$ is an order, and $(k, j)$ is a disorder. So there is no change in parity between $\sigma$ and $\rho$ for such pairs as the net change in number of disorders is zero.

In the second case, we let $\rho_k > \rho_i$ and $\rho_k < \rho_j$: In this case, in $\rho$, $(i, k)$ is an order and $(k, j)$ is also an order. In $\sigma$, $(i, k)$ is a disorder and $(k, j)$ is also a disorder. Sine 2 disorders are added, this does not change the parity of the number of disorders.

In the third case, we let $\rho_k < \rho_i$ and $\rho_k > \rho_j$: In this case, in $\rho$, $(i, k)$ is a disorder and $(k, j)$ is also a disorder. In $\sigma$, $(i, k)$ becomes an order and $(k, j)$ also becomes an order. Thus, since the net change in disorders is 2, these pairs do not cause a change in parity from the number of disorders between $\sigma$ and $\rho$.

In the last case, we let $\rho_k > \rho_i$ and $\rho_k > \rho_j$: In this case, in $\rho$, $(i, k)$ is an order and $(k, j)$ is a disorder. In $\sigma$, $(i, k)$ is a disorder and $(k, j)$ is an order. Thus these types of pairs do not change the number of disorders between $\sigma$ and $\rho$.

Thus, from this we can conclude that if the number of disorders in the subset of X for $\rho$ is $k$, then the number of disorders in the subset of $X$ for $\sigma$ will be $k+2m-2n = k+2(m-n)$ where $m$ is the number of pairs of the type discussed in case 2, and $n$ is the number of pairs of the type discussed in case 3. Thus, since we are adding an even number to $k$, the parity of $k+2(m-n)$ will be the same as the parity of $k$.

## Problem 5

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## Problem 7

a) Let's show that conjugacy is an equivalence relation. First, we will show that it is reflexive. For any $a$, choose $g = e$. Then $a = gag^{-1} = eae$ and thus $a \sim a$.

Second, we will show that it is symmetric. Let $a \sim b$. Then $b = gag^{-1}$ for some $g \in G$. Then:

$$b = gag^{-1}$$
$$g^{-1}b = ag^{-1}$$
$$g^{-1}bg = a$$

Let $x = g^{-1}$, then:

$$xbx^{-1} = a$$

Since $x$ is an element in $G$, $b \sim a$.

Last, we will show that it is transitive. Let $a \sim b$ and $b \sim c$. Then $b = g_1ag_1^{-1}$ and $c = g_2bg_2^{-1}$. Then:

$$c = g_2bg_2^{-1}$$

$$c = g_2 g_1 a g_1^{-1} g_2^{-1}$$

We observe that $(g_2 g_1)(g_1^{-1} g_2^{-1}) = e$. So $g_1^{-1} g_2^{-1} = (g_2 g_1)^{-1}$ Let $x = g_2 g_1$:

$$c = xax^{-1}$$

Since $x \in G$, $a \sim c$.

# Problem 8

a) Let's represent a point $(x, y) \in \mathbb{R}^2$ in polar coordinates as $(rcos(\theta), rsin(\theta))$. Without loss of generality let $O = (0, 0)$ and let the line be $y = 0$.

Choose an arbitrary point $p$, such that $r = r_i, \theta = \theta_i$.

We now compute $SRS^{-1}$.

After $S^{-1}$, $p$ becomes $(r_i cos(-\theta_i), r_i sin(-\theta_i))$.

After $R$, $p$ becomes $(r_i cos(-\theta_i + \alpha), r_i sin(-\theta_i + \alpha))$.

After $S$, $p$ becomes $(r_i cos(-(-\theta_i + \alpha) + 2\alpha, r_i sin(-(-\theta + \alpha) + 2\alpha))$.

Simplifying, $p$ becomes $(r_i(\theta_i + \alpha), r_i(\theta_i + \alpha))$ which is just $R$, the CCW rotation by $\alpha$.

b) We again represent $p = (x, y)$ and the origin and line the same way.

We now compute $RSR^{-1}$.

After $R^{-1}$, $p$ becomes $(r_i cos(\theta_i - \alpha), r_i sin(\theta_i - \alpha))$.

After $S$, $p$ becomes $(r_i cos(-(\theta_i - \alpha) + (-2\alpha)), r_i cos(-(\theta_i - \alpha) + (-2\alpha)))$.

After $R$, $p$ becomes $(r_i cos((-(\theta_i - \alpha) + (-2\alpha)) + \alpha), r_i cos((-(\theta_i - \alpha) + (-2\alpha))) + \alpha)$.

Simplifying, $p$ becomes $(r_i cos(-\theta_i), r_i sin(-\theta_i))$, which is just $S$, the reflection about the origin.

# Problem 9

a) Let's prove that this is a subgroup. First, we must show that $e \in Z(G)$. For any $y \in G$,

$$ey = ye$$

$$y = y$$

so $e \in G$.

Second, we must show that if $x \in Z(G)$, this implies that $x^{-1} \in Z(G)$. Let $x \in Z(G)$. Then:

$$xy = yx$$

for any $y \in G$. But then:

$$xyx^{-1} = yxx^{-1} = y$$

$$yx^{-1} = x^{-1}xyx^{-1} = x^{-1}y$$

so clearly $x^{-1}$ also in $Z(G)$.

Finally we must show closure. Let $x_1, x_2 \in Z(G)$. Then we must show that:

$$(x_1x_2)y = y(x_1x_2)$$

for any $y \in G$. Multiplying on both sides by $x_1^{-1}$ yields:

$$x_1^{-1}(x_1x_2)y = x_1^{-1}y(x_1x_2)$$

but since $x_1^{-1}$ also is in $Z(G)$, this becomes:

$$x_1^{-1}(x_1x_2)y = yx^{-1}(x_1x_2)$$

which yields

$$x_2y = yx_2$$

which we know to be true as we had $x_2 \in Z(G)$.

Thus $Z(G)$ is a subgroup.

b)