411 Individual HW8

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December 2023

Problem 1

We compute the orders of $|Fix(d)|$ for all $d \in D_6$ and then apply Burnsides lemma to find the number of orbits, which is the number of necklaces invariant under rotation and reflection. We see initially that there are 3^6 necklaces, as there are 6 jewels that can each be chosen 3 ways.

Obviously,

$$|Fix(Id)| = 3^6$$

For a rotation by 60 degrees, we see that all jewels must be the same color, otherwise the necklace will change under the reflection at at least one of the jewels. There are 3 ways to do that, because there are 3 colors.

$$|Fix(Rot_{60})| = 3$$

and we also have

$$|Fix(Rot_{300})| = 3$$

by symmetry. For a rotation by 120 degrees, we have that jewels offset by 2 from eachother must be identical. This yields us 2 sets of jewels, jewels $1, 3, 5$ and jewels $2, 4, 6$ that must be identical. There are 3 ways to choose the color of each of these sets. Thus we have

$$|Fix(Rot_{120})| = 9$$

and also

$$|Fix(Rot_{240})| = 9$$

by symmetry. For

$$|Fix(Rot_{180})|$$

, we have 3 sets of jewels, jewels 1 and 4, jewels 2 and 5, and jewels 3 and 6, which will be moved to eachother by a 180 rotation and must be identical. There are 3 ways to choose each set of these jewels.

$$|Fix(Rot_{180})| = 27$$

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There are 3 reflections about 2 jewels. For any such reflection, the two jewels on the line of reflection can each be chosen in 3 ways, and then there will be two pairs of jewels which must be the same color, and each pair can be chosen in 3 ways. So we have $|Fix(Refl_a)| = 81$.

There are 3 reflections which mirror 3 sets of jewels. Since these reflections exchange 3 pairs of jewels that must be the same and there are 3 ways to choose the color of each pair, $|Fix(Refl_b)| = 27$. We now apply Burnside's lemma and find that there are

$$\frac{3^6 + 3 + 3 + 9 + 9 + 27 + 3 \cdot 81 + 3 \cdot 27}{12} = 92$$

orbits, or 92 necklaces which are unique under rotations and reflections.

Problem 2

We define our group action to be the rotations acting on the pairs of opposite faces in the dodecahedron. There are 6 such pairs, and any of them can be moved to any other by a rotation. So this group action has one orbit of order 6.

Let's enumerate the rotations of the dodecahedron. There is 1 identity rotation, 4 rotations by 72, 144, 216, and 288 degrees about each of the 6 pairs of faces, 1 rotation by 180 degrees about each of the 15 pairs of opposite edges, and 2 rotations by 120 and 240 degrees about each of the 10 pairs of opposite vertices. This yields $1 + 4 \cdot 6 + 15 \cdot 1 + 10 \cdot 2 = 60$ rotations. We can confirm that we have found all the possible rotations because any pair of opposite rotations is stabilized by the identity, the 4 face rotations about the two faces, and the 5 edge rotations about the edges that are not adjacent to either face. Thus, the order of any stabilizer of a pair of faces is 10, and since the order of any orbit of a pair of faces is 6, we have that the order of the group is $6 \cdot 10 = 60$, and so we have confirmed that we have found every rotation of the dodecahedron.

Problem 3

a) This is an immediate consequence of Burnside's lemma. A transitive group action is a group action with only one orbit. If $|G| = \sum_{g \in G} |Fix(g)|$, then $1 = \frac{\sum_{g \in G} |Fix(g)|}{|G|}$, which tells us that the action has only 1 orbit, and that it is transitive. If the orbit is transitive, we have that $\frac{\sum_{g \in G} |Fix(g)|}{|G|} = 1$, which then yields $\sum_{g \in G} |Fix(g)| = |G|$.

b) If the group is transitive, we have

$$|G| = \sum_{g \in G} |Fix(g)|$$

Since identity fixes every element in X, we have that:

$$|G| = |X| + \sum_{g \in G, g \neq e} |Fix(g)|$$

$$|G| - |X| = \sum_{g \in G, g \neq e} |Fix(g)|$$

If every non identity element in G fixed at least 1 element, we would have that the right side of the equation is at least $|G| - 1$. But since $|X| \geq 2$, the left side is at most $|G| - 2$. So if there is no element in G that fixes no elements in the set, we would reach a contradiction. From this we have our result.

c) This is equivalent to the number of permutations which sent no element to itself. This can happen in two ways, a 5-cycle, or pair of 3-cycle and 2-cycle. There are $5!/5 = 24$ 5-cycles, and $5!/(3 \cdot 2) = 20$ 3-2-cycles, giving 44 derangements.

Problem 4

We see that there are 2^{12} ways to color the dodecahedron with two colors. There are $6 \cdot 4$ ways to rotate about the axis between two faces. There are 2 ways to color the top face, and 2 ways to color the bottom face. Also, for any rotation between 1 and 4 turns, there are no "sub-cycles" of faces that get mapped to eachother as 5 is prime. So then the top 5 and bottom 5 pentagons of the cube must be colored 1 of 2 ways. So each face rotation fixes 2^4 colorings.

For a vertex rotation, we have 4 groups of 3 pentagons that get moved to eachother by 120 and 240 rotations. Each of these can be chosen 2 ways. So a vertex rotation also fixes 2^4 colorings.

For an edge rotation, we have 6 groups of 2 pentagons that get moved to eachother. Each of these pairs can be colored in 2 ways. So an edge rotation fixes 2^6 colorings.

The identity fixes 2^{12} colorings.

We have 24 face rotations, 15 edge rotations, and 20 vertex rotations, along with the identity. Then, by Burnside lemma

$$\frac{24 \cdot 2^4 + 15 \cdot 2^6 + 20 \cdot 2^4 + 2^{12}}{60} = 96$$

and so there are 96 unique 2-colorings of the dodecahedron under rotation.

Problem 5

We can apply Burnside's lemma here, where the group action is $\mathbb{Z}_{12}$ acting on distinct triples of $\mathbb{Z}_{12}$ by component-wise modular addition.

There are $\binom{12}{3}$ total triples, as each triad is choosing 3 distinct notes without regard to order. Since triads that are shifts of eachother are equivalent, the number of distinct triads is the number of orbits yielded by this group action.

$$Fix(0) = \binom{12}{3}$$

We also have:

$$Fix(4) = 4$$

This is because for a group of 3 notes which each are 4 notes away from eachother, shifting by 4 will send them to eachother yielding the same triad. Such a triad has 4 unique start positions, and this is how we derive $Fix(4) = 4$.

An equivalent result arises for $Fix(8)$, sending 3 notes 4 notes away from eachother to themselves.

$Fix(x)$ is zero for any other $x \in \mathbb{Z}_{12}$.

Applying Burnside's lemma, we have

$$\frac{\binom{12}{3} + 4 + 4}{12} = 19$$

giving us 19 orbits, or unique triads under shifts.

Problem 6

Let R be the rotation by θ. Since the order of R is n, we have that $n\theta = 360k$ for some $k \geq 1$. We also know that $gcd(k, n) = 1$, as if k and n had a common divisor $d > 1$, we would have $\frac{n}{d}\theta = 360\frac{k}{d}$ which would imply that the order of R is less than n, which is a contradiction. We then observe that $km = 1 mod n$ for some m, as we have the existence of this modular inverse from the fact that $gcd(k, n) = 1$.

We have that

$$\theta = \frac{360k}{n}$$

If $k = 1$, then $\theta = \frac{360}{n}$, and we are done. Otherwise, we have that

$$m\theta = \frac{360km}{n}$$

Since $km = 1 \ mod \ n$

$$m\theta \equiv \frac{360(\ell n + 1)}{n} = 360\ell + \frac{360}{n} \equiv \frac{360}{n}$$

for some ℓ. But this is equivalent to the rotation R^m and $R^m \in \langle R \rangle$. So we are done.

Problem 7

Take arbitrary $x, y \in \mathbb{R}^2$. Let $T(x) = x', T(y) = y'$. Then $T^{-1}(x') = x, T^{-1}(y') = y$. Any isometry preserves both the distance and angle between two points in the domain in its image. From this we have that the angle between x and y is the same as the angle between x' and y'. Let R move x' to y'. We compute then that $TRT^{-1}(x') = y'$, which is the same for R for any arbitrary points. So it is shown. We also now have that the center of R is $Isom(\mathbb{R}^2)$ itself, as $gRg^{-1} = R$ for any $g \in \text{Isom}\mathbb{R}^2$.

Problem 8

a) Conjugacy classes are equivalent to orbits of the group action of conjugation of a group by itself. By the orbit stabilizer thoerem, the orders of these conjugacy classes will divide the order of the group.

b) If G is Abelian, we are done. Otherwise, $Z(G)$ is a proper subgroup of G. The order of $Z(G)$ is then at most $\frac{|G|}{2}$ by Lagrange's theorem. Every element in the center constitutes its own conjugacy class. We then have $\frac{|G|}{2}$ other elements, which are divided into conjugacy classes of size at least 2. These elements can then be divided into at most $\frac{|G|}{4}$ conjugacy classes. Adding these numbers, we see that a non abelian group G has at most $\frac{|G|}{2} + \frac{|G|}{4} = \frac{3|G|}{4}$ conjugacy classes.

Problem 9

We know that a group action partitions a set into distinct orbits. Thus:

$$|X| = \sum_{X_i \subseteq X} |X_i|$$

We see that $|X_i|$ must be some power of p, by the Orbit-stabilizer theorem and Lagrange's theorem, as stabilizers must be powers of p as they are subgroups and must divide the order of $|G|$ which is p^k. An power of p divided by a power of p yields a power of p. We turn to the elements in question, whose orbit is only itself, and therefor its stabilizer the whole group. Thus, we see we can factor out p from the order of the orbit of any element which is not stabilized by the whole group. We then have

$$|X| = 1 + \cdots + 1 + pk$$

for some $k \in \mathbb{Z}$. The number of 1s will be the number of elements who are stabizlied by the whole group. Thus, we have shown that $|X|$ is congruent to this number mod p.