Analysis Problem Set 1

Jack Madden

May 2024

Exercise 1.3

Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

For the base case, it is easy to see that $1^3 = 1^2$. Inductively, suppose that this holds for $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$. Then:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \stackrel{?}{=} (1 + 2 + \cdots + n + n + 1)^2 = ((1 + 2 + \cdots + n) + (n+1))^2$$

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \stackrel{?}{=} (1 + 2 + \cdots + n)^2 + 2(n+1)(1 + 2 + \cdots + n) + (n+1)^2$$

By our inductive hypothesis we have:

$$(n+1)^3 \stackrel{?}{=} 2(n+1)(1 + 2 + \cdots + n) + (n+1)^2$$

Employing

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$(n+1)^3 \stackrel{?}{=} n(n+1)^2 + (n+1)^2$$

Factoring yields:

$$(n+1)^3 = (n+1)^2(n+1) = (n+1)^3$$

So this equality holds and thus we have:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for all $n \in \mathbb{N}$.

Exercise 1.8

a) *Prove that* $n^2 > n + 1$ *for* $n \geq 2$

For our base case we have that $4 = 2^2 > 2 + 1 = 3$. Suppose inductively that this holds for n. Then we have that

$$(n+1)^2 \stackrel{?}{>} (n+1) + 1$$

$$n^2 + 2n + 1 \stackrel{?}{>} (n+1) + 1$$

Inductively we have:

$$2n > 0$$

and since n is positive this inequality clearly holds.

b) *Prove $n! > n^2$ for $n \geq 4$*

For our base case we have that $24 = 4! > 4^2 = 16$. Consider the inequality

$$(n+1)! > (n+1)^2$$

$$(n+1)! > n^2 + 2n + 1$$

Since $n \geq 4$, and by our inductive hypothesis we have that

$$(n+1)! = (n+1)n! > 3n! > n^2 + 2n + 1$$

and since $n \geq 4 > 3$, this inequality holds.

Exercise 1.9

Show that $2^n > n^2$ for $n \geq 5$

For the base case we have $32 = 2^5 > 5^2 = 25$. Proceeding inductively:

$$2^{n+1} \overset{?}{>} (n+1)^2 = n^2 + 2n + 1$$

By our inductive hypothesis we have:

$$2^{n+1} \overset{?}{>} 2^n + 2^{\frac{n}{2}+1} + 2^0 > n^2 + 2n + 1$$

Factoring gives:

$$2^{n+1} \overset{?}{>} 2^n \left(1 + \frac{1}{2^{\frac{n}{2}-1}} + \frac{1}{2^n}\right) > n^2 + 2n + 1 = (n+1)^2$$

Since $n \geq 5$

$$2^{n+1} = 2 \cdot 2^n = (1 + \frac{1}{2} + \frac{1}{2}) > 2^n \left(1 + \frac{1}{2^{1.5}} + \frac{1}{2^5}\right) > n^2 + 2n + 1 = (n+1)^2$$

and this relation holds.

Exercise 2.3

Suppose that $\sqrt{2+\sqrt{2}}$ is rational. We have that

$$\sqrt{2+\sqrt{2}} = x$$

$$\sqrt{2} = x^2 - 2$$

$$2 = x^4 - 4x^2 + 4$$

so then this is a zero of the polynomial:

$$f(x) = x^4 - 4x^2 + 2$$

If this polynomial has rational zeroes, they are either $\pm 1, \pm 2$. But $f(\pm 1) = -1$, and $f(\pm 2) = 2$. So since this polynomial has no rational zeroes, $\sqrt{2+\sqrt{2}}$ cannot be rational.

Exercise 2.4

Suppose that $\sqrt[3]{5-\sqrt{3}}$ is rational. Then

$$x = \sqrt[3]{5-\sqrt{3}}$$
$$x^3 - 5 = -\sqrt{3}$$
$$x^6 - 10x^3 + 25 = 3$$

and $\sqrt[3]{5-\sqrt{3}}$ is a zero of $f(x) = x^6 - 10x^3 - 22$ Then the rational zeroes of this polynomial must $\pm 1, \pm 2, \pm 11, \pm 22$. We have $f(1) = -31, f(-1) = -11$. Also,

$$f(2) = 2(2^5 - 10 \cdot 2^2 - 11)$$

and since $(2^5 - 10 \cdot 2^2 - 11)$ is odd it is not zero and hence $f(2) \neq 0$. $f(-2) \neq 0$ by the same argument. Also,

$$f(11) = 11^6 - 10 \cdot 11^3 - 22$$
$$f(11) = 11^6 - 2(5 \cdot 11^3 - 11)$$

Observe that $-2(5 \cdot 11^3 - 11)$ is even and 11^6 is odd. Thus $f(11)$ is odd and therefore not zero. A similar argument follows for $f(-11)$. Also,

$$f(22) = 22^6 - 10 \cdot 22^3 - 22 = 22(22^5 - 10 \cdot 22^2 - 11)$$

Observe that $22^5 - 10 \cdot 22^2 - 11$ is odd and hence not zero. Thus the product of 22 and something that is not zero cannot be zero and we have that $f(\pm 22) \neq 0$. Hence, this polynomial has no rational roots and thus $\sqrt[3]{5-\sqrt{3}}$ is irrational.

Exercise 2.7

a) Let $x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$. Then

$$x + \sqrt{3} = \sqrt{4+2\sqrt{3}}$$
$$x^2 + 2\sqrt{3}x + 3 = 4 + 2\sqrt{3}$$

Observe that this holds for $x = 1$. Thus $x = \sqrt{4+2\sqrt{3}} - \sqrt{3} = 1 \in \mathbb{Q}$ and this number is rational.

b) Let $x = \sqrt{6+4\sqrt{2}} - \sqrt{2}$. Then

$$(x + \sqrt{2})^2 = 6 + 4\sqrt{2}$$
$$x^2 + 2\sqrt{2}x + 2 = 6 + 4\sqrt{2}$$

Observe that this holds for $x = 2$, and thus $\sqrt{6+4\sqrt{2}} - \sqrt{2} = 2 \in \mathbb{Q}$.