Analysis Problem Set 2

Jack Madden

May 2024

Exercise 4.5

Let S be a non-empty subset of $\mathbb{R}$ that is bounded above. Suppose $s_0 = sup\ S \in S$. Then $s_0 \geq s$ for all $s \in S$, and $s_0 \in S$, and thus by definition $s_0 = sup\ S = max\ S$.

Exercise 4.10

By the Archimedean property, we have that there is some $k \in \mathbb{N}$ such that $1 < ak \rightarrow \frac{1}{k} < a$, and that there is some $m \in \mathbb{N}$ such that $a < m$. We claim then that the following inequality holds:

$$\frac{1}{max(k,m)} < a < max(k,m)$$

Suppose that $k < m$. By Theorem 3.2, we have that $\frac{1}{m} < \frac{1}{k} < a$, which means that $\frac{1}{m} < a < m$ holds. Likewise, suppose $m < k$. Then clearly $a < m < k$ which means that $\frac{1}{k} < a < m < k$ holds, or that $\frac{1}{k} < a < k$ holds. Thus, there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Exercise 4.12

We first show that $r + \sqrt{2} \in \mathbb{I}$ for any $r \in \mathbb{Q}$. Suppose $s = r + \sqrt{2}$ for some $r, s \in \mathbb{Q}$. But then, $s - r = \sqrt{2}$. Since $r, s \in \mathbb{Q}$, $s - r \in \mathbb{Q}$ which implies $\sqrt{2} \in \mathbb{Q}$ which is a contradiction. Hence $r + \sqrt{2} \in \mathbb{I}$ for all $r \in \mathbb{Q}$. Consider the inequality $a - \sqrt{2} < x < b - \sqrt{2}$, for any $a, b \in \mathbb{R}, a < b$. We know that this holds for some $x \in \mathbb{Q}$ due to the denseness of $\mathbb{Q}$. Therefore, by adding $\sqrt{2}$ to the inequality we get that $a < x + \sqrt{2} < b$ holds for any a, b, and since $x + \sqrt{2} \in \mathbb{I}$, we know that for any a, b there exists some $y \in \mathbb{I}$ such that $a < y < b$.

Exercise 4.15

We show the contrapositive, i.e. $a > b$ implies that there exists some $n \in \mathbb{N}$ such that $a > b + \frac{1}{n}$.

$$a > b + \frac{1}{n}$$

$$an > bn + 1$$

$$(a - b)n > 1$$

Since $a > b$, $a - b > 0$, and hence by the Archimedean property, there exists some n which satisfies this equation.